BERRY PHASE WITH ENVIRONMENT: CLASSICAL VERSUS QUANTUM

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Abstract We discuss the concept of the Berry phase in a dissipative system. We show that one can identify a Berry phase in a weakly-dissipative system and find the respective correction to this quantity, induced by the environment. This correction is expressed in terms of the symmetrized noise power and is therefore insensitive to the nature of the noise representing the environment, namely whether it is classical or quantum mechanical. It is only the spectrum of the noise which counts. We analyze a model of a spin-half (qubit) anisotropically coupled to its environment and explicitly show the coincidence between the effect of a quantum environment and a classical one.

Keywords: adiabaticity, Berry phase, dissipative dynamics, Lamb shift

Introduction

Three papers published independently in 1932 by Zener, Landau and Stueckelberg (Landau, 1932; Zener, 1932; Stueckelberg, 1932) have introduced the phenomenon known today as Landau-Zener tunneling. The idea is to consider a 2-level system, where the energy of each level varies linearly with a classical variable (which, in turn, is varied linearly in time). As function of time, t, the energy levels should intersect but for the inter-level coupling Δ which gives rise to an "avoided crossing" in the spectrum, cf. Fig.1. Using the spin notation, one can write the Hamiltonian as $\hat{\mathcal{H}} = \alpha t S_z + \Delta S_x$. Here $\boldsymbol{S} = \boldsymbol{\sigma}/2$, and σ_z, σ_x are Pauli spin-1/2 operators; α is the rate of change of the energy of the pseudo-spin at asymptotic times. The avoided crossing gap is Δ. The probability of transition from, say, the lower level at time $-\infty$, to the upper level at time $+\infty$ is given by $P_{\mathrm{LZ}} = \exp[-(\pi/2)\Delta^2/\alpha]$.

Besides being ubiquitous in physics and chemistry, the Landau-Zener framework appears to suggest a natural definition for the notion of adiabaticity. The adiabatic limit is approached when $P_{\mathrm{LZ}} << 1$, i.e., $\alpha << \Delta^2$. The latter inequality involves a comparison of the rate of change (of the time dependent term in the Hamiltonian) with the gap in the spectrum, Δ. This notion of the adiabatic limit has become widespread. A closer look suggests that, in general, adiabaticity **cannot** be associated with comparing the rate of change to the gap. Indeed, on one hand any finite, discrete-spectrum system is coupled, however weakly, to the rest of the universe. Hence the emerging spectrum is, at least in principle, always continuous and gapless. The naive view would then imply that the adiabatic limit cannot be approached. This, on the other hand cannot be correct: if we consider a finite system with a discrete spectrum, for which adiabaticity is well defined, it is inconceivable that an infinitesimal coupling to the continuum (rendering the overall spectrum continuous) will change its physics in a dramatic way. The resolution of this problem is provided by the observation that the criterion for adiabaticity involves not only spectral properties but also the **matrix elements** of the system-environment coupling.

To gain some insight into this problem we focus here on the analysis of the Berry phase (Berry, 1984) in a weakly dissipative system. It is particularly timely to address this issue now given the recent experimental activities in realization of controlled quantum two-level systems (qubits), and in particular, the interest in observing a Berry phase (BP) (see, e.g., (Falci et al., 2000)). For instance, the superconducting qubits have a coupling to their environment, which is weak but not negligible (Nakamura et al., 1999; Vion et al., 2002; Chiorescu et al., 2002),

and thus it is important to find both the conditions under which the Berry phase can be observed and the nature of that Berry phase.

In this paper we appeal to a simple analysis of the problem. We first, in Section 2, consider a quantum-mechanical framework, where a perturbative approach is taken. When the environment is replaced by a single oscillator, a second-order perturbation analysis is straightforward and produces a result which allows for a simple interpretation. We then generalize the calculation for a host of environmental modes. In Section 3 we consider a toy model where the environment is replaced by a **classical** stochastic force. The quantities of interest, the Lamb shift and the Berry phase, are then calculated, and simple heuristic arguments are given to interpret the results. To complete the analogy with the analysis of the previous section, here the "single-oscillator environment" is replaced by a simple periodic classical force (of random amplitude). In Section 4 we summarize the relation between the quantum mechanical approach and the classical model in more general terms.

1. The system: spin + environment

We begin in the conventional way by writing the Hamiltonian for the "universe" (system + environment) as

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\text{syst}} + \hat{\mathcal{H}}_{\text{env}} + \hat{\mathcal{V}}_{\text{coupling}} \tag{1}$$

The system is defined as the set of those quantum degrees of freedom that one is interested to control and measure; the environment consists of all the rest, namely those degrees-of-freedom we can neither control nor measure. The coupling between the system and environment is $\mathcal{V}_{\text{coupling}}$. The properties of the environment are controlled by macroscopic parameters, such as temperature. Our treatment below applies to a reservoir at either zero or a finite temperature.

For our purposes it is sufficient to represent the environment by a single operator X which couples to a spin. The Hamiltonian then becomes

$$\hat{\mathcal{H}} = -\tfrac{1}{2}\,\mu g\,\boldsymbol{B}\cdot\hat{\boldsymbol{\sigma}} - \tfrac{1}{2}X\sigma_z + \hat{\mathcal{H}}_{\text{env}}\,. \tag{2}$$

Hereafter we put $\mu g = 1$. Below we express our results in terms of the statistical properties (correlators) of the environment's noise, $X(t)$. Depending on the physical situation at hand, one can choose to model the environment via a bath of harmonic oscillators (Feynman and Vernon, 1963; Caldeira and Leggett, 1983). In this case the generalized coordinate of the reservoir is defined as $X = \sum \lambda_i x_i$, where $\{x_i\}$ are the coordinate operators of the oscillators and $\{\lambda_i\}$ are the respective couplings. Eq. 2 is then referred to as the spin-boson Hamiltonian (Leggett et al.,

1987). Another example of a reservoir could be a spin bath (Prokof'ev and Stamp, 2000) [1]. However, in our analysis below we do not specify the type of the environment. We will only assume that the reservoir gives rise to markovian evolution on the time scales of interest. More specifically, the evolution is markovian at time scales longer than a certain characteristic time τ_c, determined by the environment [2]. We assume that τ_c is shorter than the dissipative time scales introduced by the environment, such as the dephasing or relaxation times and the inverse Lamb shift (the scale of the shortest of which we denote as $T_{\rm diss}$, $\tau_c \ll T_{\rm diss}$). We further assume that $\tau_c \ll t_{\rm P}$, the characteristic variation time of the field $\mathbf{B}(t)$. Moreover, under these conditions we may consider only lowest-order (in the system-environment coupling) contributions to the quantities of interest: energy shifts, BP and relaxation rates. Indeed, if one divides the evolution time interval into short domains ($\ll t_{\rm P}$), longer than τ_c but shorter than $T_{\rm diss}$, fluctuations at different domains are uncorrelated and their effect can be analyzed separately. At the same time, for each domain ($\ll T_{\rm diss}$) the effect of the noise is weak. Thus, to the leading order corrections to the dynamics may be described as corrections to the rates (energies) of the spin dynamics, which may be estimated perturbatively. We also consider an underdamped spin, with the dissipative times longer than the period of the coherent dynamics, $T_{\rm diss} \gg 1/B$. This implies that the time windows alluded to above consist of numerous oscillations, in other words they are $\gg 1/B$.

We have chosen an anisotropic spin-environment coupling, $\propto \sigma_z$. This is a realistic model, e.g., for many designs of solid-state qubits, where the different components of the "spin" are influenced by entirely different environmental degrees of freedom (Nakamura et al., 1999; Vion et al., 2002; Chiorescu et al., 2002). While our analysis can be generalized to account for multiple-directional fluctuating fields (Whitney et al., 2004), here we focus on unidirectional fluctuations (along the z axis).

[1] For any reservoir in equilibrium the fluctuation-dissipation theorem provides the relation between the symmetrized and antisymmetrized correlators of the noise: $S_X(\omega) = A_X(\omega)\coth(\omega/2T)$. Yet, the temperature dependence of S_X and A_X may vary depending on the type of the environment. For an oscillator bath, A_X (also called the spectral density $J_X(\omega)$) is temperature-independent, so that $S_X(\omega) = J_X(\omega)\coth(\omega/2T)$. On the other hand, for a spin bath S_X is temperature-independent and is related to the spins' density of states, while $A_X(\omega) = S_X(\omega)\tanh(\omega/2T)$.

[2] This time may be given by the correlation time of the fluctuations, but in general is a more subtle characteristic of the spectrum related to its roughness near qubit's frequencies. Note further that for singular spectra τ_c may be ill defined and the perturbative analysis may fail. See, e.g., (Bloch, 1957; Redfield, 1957; Slichter, 1978; Makhlin et al., 2003; Wilhelm et al., 2004; Whitney and Gefen, 2004).

Another remark to be made concerns the possibility to observe a (weak) dissipative correction to Berry phase in spite of the dephasing and relaxation phenomena. While the respective time scales (T_1, T_2 and the inverse of the correction to the Berry phase) scale similarly with the strength of fluctuations (inversely proportionally to the noise power), they are dominated by different frequency domains. Indeed, the dephasing and relaxation are known to be dominated by resonant fluctuations with frequencies close to B (for the relaxation and the corresponding contribution to dephasing) and 0 (for the pure dephasing), cf. Eq. (15) below. In contrast, as we shall see below, the Lamb shift and the correction to the Berry phase accumulate contribution from the entire range of frequencies. Thus, one may think of (engineering) a system with an environment whose fluctuations at $\nu \sim B$ and $\nu \sim 0$ are suppressed. In this case, one can easily observe an observable correction to the Berry phase at times when the dephasing and relaxation are still negligible.

2. Quantum-mechanical analysis

In this section we consider a two-level system coupled to an environment which we treat as a quantum-mechanical system. We begin with a discussion of the Lamb shift and then show, in Subsection 2.3, how the results for the Lamb shift may be used to find the environment-induced correction to the Berry phase and the relaxation times.

2.1 Lamb shift as level repulsion

Consider first, for illustration, a simple system of the spin coupled to a single oscillator, with the Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}B\sigma_z - \tfrac{1}{2}c\sigma_x(a^\dagger + a) + \omega_0 a^\dagger a \ , \tag{3}$$

where c is the coupling constant. Let $|n\rangle$ denote the n-th level of the oscillator; the second-order corrections to the energies of the states $|\uparrow, 0\rangle$ and $|\downarrow, 0\rangle$ are

$$E_\uparrow^{(2)} = -\frac{|\langle\uparrow, 0|\,\mathcal{V}\,|\downarrow, 1\rangle|^2}{\omega_0 + B} = -\frac{1}{4}\,\frac{c^2}{\omega_0 + B}\,, \tag{4}$$

and

$$E_\downarrow^{(2)} = -\frac{|\langle\downarrow, 0|\,\mathcal{V}\,|\uparrow, 1\rangle|^2}{\omega_0 - B} = -\frac{1}{4}\,\frac{c^2}{\omega_0 - B}\,, \tag{5}$$

where $\mathcal{V} \equiv (c/2)\,\sigma_x\,(a^\dagger + a)$ is the perturbation. This results in the following correction to the level spacing $E_\downarrow - E_\uparrow$:

$$E_\downarrow^{(2)} - E_\uparrow^{(2)} = \frac{c^2}{2}\frac{B}{B^2 - \omega_0^2}\,. \tag{6}$$

This correction (the Lamb shift) has different signs for fast ($\omega_0 > B$) and slow ($\omega_0 < B$) oscillators. As one can see from Eqs. (4), (5), this result can be understood in terms of the level repulsion (Wilhelm et al., 2004): the perturbation couples the level $|\uparrow, 0\rangle$ to $|\downarrow, 1\rangle$ and $|\downarrow, 0\rangle$ to $|\uparrow, 1\rangle$. The levels of the latter pair are closer, and the coupling has a stronger effect on their energies. They repel each other due to the coupling, thus reducing the distance between $|\uparrow, 0\rangle$ and $|\downarrow, 0\rangle$ for $\omega_0 > B$ and increasing it for $\omega_0 < B$.

2.2 Second-order perturbative analysis

In this section we find the Lamb shift using the lowest-, second-order perturbative analysis. In the Hamiltonian (2) we treat the coupling term $\mathcal{V} = -\frac{1}{2}X\sigma_z$ as a perturbation: $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$. The eigenstates of $\mathcal{H}_0$ are $|\alpha, i\rangle$, where $\alpha = \uparrow_B/\downarrow_B$ denotes the eigenstates of the spin without dissipation, with the spin direction parallel or antiparallel to the filed $\boldsymbol{B}$, and i denotes eigenstates of the environment. The perturbation theory gives for the corrections to their eigenenergies:

$$E^{(2)}_{\alpha,i} = -\sum_{\beta,j} \frac{|\langle\alpha, i|\, \mathcal{V}\, |\beta, j\rangle|^2}{E^{(0)}_\beta + E^{(0)}_j - E^{(0)}_\alpha - E^{(0)}_i - \mathrm{i}0}\,. \tag{7}$$

For $\mathcal{V} = -\frac{1}{2}X\sigma_z$ we notice that $\langle\uparrow_B|\,\sigma_z\,|\uparrow_B\rangle^2 = \langle\downarrow_B|\,\sigma_z\,|\downarrow_B\rangle^2 = \cos^2\theta$ and $\langle\uparrow_B|\,\sigma_z\,|\downarrow_B\rangle^2 = \langle\downarrow_B|\,\sigma_z\,|\uparrow_B\rangle^2 = \sin^2\theta$, and find for the environment-averaged quantities $E^{(2)}_\alpha \equiv \sum_i \rho_i\, E^{(2)}_{\alpha,i}$ (see the discussion of these quantities at the end of this subsection):

$$E^{(2)}_\uparrow = -\frac{\cos^2\theta}{4}\sum_{i,j} \frac{\rho_i\,|\,\langle i|\,X\,|j\rangle\,|^2}{E^{(0)}_j - E^{(0)}_i - \mathrm{i}0} - \frac{\sin^2\theta}{4}\sum_{i,j} \frac{\rho_i\,|\,\langle i|\,X\,|j\rangle\,|^2}{B + E^{(0)}_j - E^{(0)}_i - \mathrm{i}0}\,. \tag{8}$$

The correction to $E_\downarrow$ is obtained by substituting $B \to -B$ into the above equation. Now using the identity

$$\frac{1}{E - \mathrm{i}0} = \mathrm{i}\int_0^\infty dt\, e^{-i(E-\mathrm{i}0)t}\,, \tag{9}$$

we rewrite Eq. (8) as

$$E^{(2)}_\uparrow = -\frac{\mathrm{i}}{4}\int_0^\infty dt\,\langle X(t)X(0)\rangle \left(\cos^2\theta + \sin^2\theta e^{-iBt}\right) e^{-0t}\,, \tag{10}$$

where we have used the relation

$$\langle X(t)X(0)\rangle = \sum_{i,j} \rho_i\,\langle i|\,X\,|j\rangle\,\langle j|\,X\,|i\rangle\, e^{-i(E_j - E_i)t}\,. \tag{11}$$

In terms of the the Fourier transform $\langle X_\nu^2 \rangle \equiv \int dt\, \langle X(t)X(0)\rangle\, e^{i\nu t}$ we obtain

$$E_\uparrow^{(2)} = -\frac{1}{4}\cos^2\theta \int \frac{d\nu}{2\pi}\frac{\langle X_\nu^2\rangle}{\nu - \mathrm{i}0} - \frac{1}{4}\sin^2\theta \int \frac{d\nu}{2\pi}\frac{\langle X_\nu^2\rangle}{\nu + B - \mathrm{i}0}\,. \qquad (12)$$

For the Lamb shift $E_{\mathrm{Lamb}}^{(2)} \equiv \Re\mathrm{e}(E_\downarrow^{(2)} - E_\uparrow^{(2)})$ this gives a principal value integral

$$E_{\mathrm{Lamb}}^{(2)} = \tfrac{1}{2}\sin^2\theta\; \mathcal{P}\!\!\int \frac{d\nu}{2\pi}\frac{S_X(\nu)}{B-\nu} = B\sin^2\theta\; \mathcal{P}\!\!\int_0^\infty \frac{d\nu}{2\pi}\frac{S_X(\nu)}{B^2-\nu^2}\,, \qquad (13)$$

where

$$S_X(\nu) \equiv \tfrac{1}{2}(\langle X_\nu^2\rangle + \langle X_{-\nu}^2\rangle) = \tfrac{1}{2}\int dt\,\langle [X(t),X(0)]_+\rangle\, e^{i\nu t}\,. \qquad (14)$$

Thus the Lamb shift is expressed in terms of the symmetrized correlator S_X and is insensitive to the antisymmetric part of the noise spectrum.

As one can see from Eq. (13), in agreement with the discussion in the previous section, the high-frequency noise ($\nu > B$) reduces the energy gap between the spin states (Leggett et al., 1987), while the low frequency modes ($\nu < B$) increase the energy gap.

Similarly, from Eq. (12) one can evaluate the dephasing time:

$$\frac{1}{T_2} = -\,\Im\mathrm{m}(E_\uparrow^{(2)} + E_\downarrow^{(2)}) = \frac{\cos^2\theta}{4}S_X(\nu=0) + \frac{\sin^2\theta}{4}S_X(\nu=B)\,. \qquad (15)$$

This expression correctly reproduces the contribution of the transverse fluctuations ($\propto \sin^2\theta$) to the dephasing rate, but underestimates the longitudinal contribution ($\propto \cos^2\theta$) by a factor of two (cf. Ref. (Bloch, 1957; Redfield, 1957; Weiss, 1999)). One can show that an accurate evaluation of this contribution, as well as the analysis of the relaxation, requires taking into account corrections to the eigenstates, and not only to the eigenenergies (7). More precisely, our calculation of the corrections to the eigenenergies in this subsection corresponds to evaluation only of the four left diagrams in Fig. 7 of Ref. (Makhlin et al., 2003); the term i0 in the denominators allows one to find also the outgoing transition rates *from* the eigenstates (and the respective contribution, $\propto \sin^2\theta$, to dephasing) but only the part of the 'pure-dephasing' rate, $\propto \frac{1}{4}\cos^2\theta$. Analysis of the two remaining diagrams in Fig. 7 and those in Fig. 6 allows one to find also the pure dephasing rate (as well as the incoming transition rates, the latter though do not require an extra evaluation due to probability conservation).

2.3 From Lamb shift to Berry phase

So far we have analyzed the environment-induced correction to the level splitting (the Lamb shift). Using the results above one can evaluate also the environment-induced correction to the Berry phase for a slow cyclic variation of the magnetic field $\boldsymbol{B}$ (Whitney and Gefen, 2001; Whitney and Gefen, 2003; Whitney et al., 2004; Whitney and Gefen, 2004).

Indeed, consider the simplest case of conic variations of the field around the z-axis (to which the environment is coupled), as shown in Fig. 1: the field varies at a constant rate, with the low angular velocity $\omega_{\rm B}$, and traverses the circle after the period $t_P \equiv 2\pi/\omega_{\rm B}$. The analysis of the spin dynamics is considerably simplified by going to the frame, rotating with the angular velocity $\omega_{\rm B}\hat{z}$, where $\hat{z}$ is the unit vector along the z-axis. In this frame the spin is subject to the fluctuating field $X\hat{z}$ and the field $\boldsymbol{B}+\omega_{\rm B}\hat{z}$, which is *stationary*. Thus, in this frame one can use the results of the analysis above to obtain the Lamb shift, if one substitutes $\boldsymbol{B}$ by $\boldsymbol{B}+\omega_{\rm B}\hat{z}$. In other words, the correction to the Lamb shift associated with the variation of the field $\boldsymbol{B}$ in time, is given by taking the derivative $\omega_{\rm B}\partial_{B_z}$ of the Lamb shift (13) and multiplying by the period of variation, t_P. After a full period the basis of the rotating frame makes a complete circle and returns to its initial position, i.e. coincides with the laboratory frame's basis. Hence the phases accumulated in the rotating and laboratory frames coincide, and it is sufficient to evaluate it in the rotating frame. Thus, one finds the environment-induced correction to the Berry phase to be

$$\delta\Phi_{BP} = 2\pi \, \frac{\partial E_{\rm Lamb}(\boldsymbol{B})}{\partial B_z} \, . \tag{16}$$

Taking the derivative of Eq. (13), we find:

$$\delta^{(2)}\Phi_{BP} = \cos\theta \sin^2\theta \; \mathcal{P}\!\!\int d\nu \, \frac{S_X(\nu)(2\nu-3B)}{2B(B-\nu)^2} \, . \tag{17}$$

(Notice the convention: this expression gives the correction to the relative Berry phase between the spin-up and spin-down states, rather than to the phases of each of these states.) As for the Lamb shift, the contributions of the high- and low-frequency fluctuations are of opposite signs. For the Berry phase the contribution changes sign at $\nu = 3B/2$.

In passing we note that this analysis can be generalized to an arbitrary (but adiabatic) path $\boldsymbol{B}(t)$, this enables one to see that the correction to the Berry phase is geometric, but that its geometric nature is very

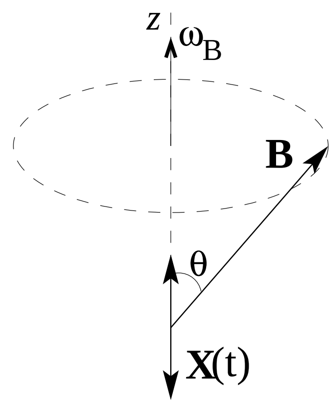

Figure 1.

different from the Berry phase of an isolated spin-half (Whitney et al., 2004).

In Section 3 we shall find exactly the same expression for the Lamb shift and therefore for the Berry phase in the case of classical environment.

2.4 High-frequency noise: renormalization of the transverse *B*-field

Consider now the influence of the high-frequency fluctuations in the environment only ($\nu \gg B$). Since the frequencies of the fluctuations are much higher than the typical spin-dynamics frequencies, one may eliminate these high-frequency fluctuations using the adiabatic (Born-Oppenheimer) approximation, as described, e.g., by Leggett et al. (Leggett et al., 1987).

Indeed, consider the spin-boson model, with the Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}(\boldsymbol{B} + X\hat{z})\boldsymbol{\sigma} + \mathcal{H}_{\rm env}\,, \tag{18}$$

where $X = \sum_i c_i (a_i^\dagger + a_i)$ and $\mathcal{H}_{\rm env} = \sum_i \omega_i a_i^\dagger a_i$. Let us ignore the low-frequency oscillators and focus on those at high frequencies $\nu \gg B$. These fast oscillators adjust almost instantaneously to the slowly varying spin state. For the last two terms of the Hamiltonian (18) two lowest-energy states are $\left|\tilde{\uparrow}\right\rangle = |\uparrow\rangle \prod_i \left|g_i^\uparrow\right\rangle$ and $\left|\tilde{\downarrow}\right\rangle = |\downarrow\rangle \prod_i \left|g_i^\downarrow\right\rangle$. Here $\left|g_i^\uparrow\right\rangle$ denotes the ground state of the ith oscillator corresponding to the spin state $|\uparrow\rangle$, i.e. the ground state of $\omega_i a_i^\dagger a_i + c_i (a_i^\dagger + a_i)$, and $\left|g_i^\downarrow\right\rangle$ is defined similarly; further eigenstates of the last two terms are separated by a gap $\sim \nu$.

Consider now the matrix elements of the first term $-\frac{1}{2}\boldsymbol{B}\boldsymbol{\sigma}$ in this two-state low-energy subspace; one finds that its transverse component is suppressed by the factor

$$\prod_i \left\langle g_i^{\uparrow} \middle| g_i^{\downarrow} \right\rangle = \prod_i \exp(-c_i^2/2\omega_i^2) = \exp\left(-\int_0^\infty \frac{d\nu}{2\pi} \frac{J_X(\nu)}{\nu^2} \right) , \tag{19}$$

where $J_X(\nu) \equiv \pi \sum_i c_i^2 \delta(\nu - \omega_i)$ is the spectral density of the oscillator bath. At a finite temperature T each high-frequency oscillator remains in its thermal equilibrium state (subject to the spin state), rather than the ground state, and on the rhs of Eq. (19) the spectral density $J_X(\nu)$ is replaced by the thermal noise power $S_X(\nu) = J_X(\nu)\coth(\nu/2k_\mathrm{B}T)$.

Thus the role of the high-frequency oscillators is to suppress the transverse field component (in other words, the transverse g-factor). If we are interested only in the contribution to the level spacing (the Lamb shift), one should consider only the longitudinal ($\parallel \boldsymbol{B}$) part of the renormalization, i.e. multiply the result by $\sin\theta$, to obtain Eq. (13).

2.5 Effective-action analysis

One can study the spin dynamics integrating out the environment and using the effective action for the spin. We derive the effective action using the Feynman-Vernon-Keldysh technique. For the interaction $-Xs_z$ with the z-component of the spin, the effective action (the influence functional) reads

$$\mathrm{i}\Phi_\mathrm{infl} = -\frac{1}{2} \int_{C_K} dt \int_{C_K} dt'\, s_z(t) \cdot s_z(t') \,[\,\mathrm{i}G_X(t,t')\,] , \tag{20}$$

where we assumed the Gaussian statistics of X, and defined the Green function G_X as $\mathrm{i}G_X(t,t') = \langle T_{C_K} X(t)X(t')\rangle$. The time ordering here refers to the Keldysh time contour C_K, and in Eq. (20) we integrate over C_K; accordingly each of the time dependent variables assumes a 'Keldysh index' u, d indicating the upper/lower branch of this contour.

After the Keldysh rotation one obtains the influence functional in terms of the classical and quantum components, $s_z^c \equiv (s_z^u + s_z^d)/2$ and $s_z^q \equiv s_z^u - s_z^d$:

$$\Phi_\mathrm{infl} = -\int dt dt' \left[s_z^q(t) G_X^\mathrm{R}(t-t') s_z^c(t') + \frac{1}{4}\, s_z^q(t) G_X^\mathrm{K}(t-t') s_z^q(t') \right] , \tag{21}$$

in terms of the retarded and Keldysh Green functions, $G_X^\mathrm{R} \equiv -\mathrm{i}\theta(t-t')\langle [X(t), X(t')]_- \rangle$ and $G_X^\mathrm{K} \equiv -\mathrm{i}\langle [X(t), X(t')]_+ \rangle = -2\mathrm{i}S_X(t-t')$.

For classical noise X the commutator in the definition of G^{R} vanishes, and one finds

$$\Phi_{\mathrm{infl}}^{\mathrm{class}} = \frac{\mathrm{i}}{2} \int dt \int dt' \; s_z^q(t)\, S_X(t-t')\, s_z^q(t')\,. \tag{22}$$

The results (13), (17) for the Lamb shift and the Berry phase involve only S_X and not the antisymmetrized correlator. Hence for the analysis of these quantities it should be sufficient to use the functional (22). Alternatively, one may consider a problem with a classical random field $\boldsymbol{X}(t)$ to reproduce these results. In the next section we perform the corresponding analysis.

3. The Classical Model

In this section we analyze the dynamics of a spin subject to a classical random field and derive the equation of motion for the spin dynamics (the spin-evolution operator), averaged over the fluctuations. Following the discussion of the case with quantum fluctuations, we first analyze the dynamics in a stationary field $\boldsymbol{B}$ and a random field; exactly as in the quantum case one can reduce the analysis of the dissipative corrections to the Berry phase accumulated over a conic loop to the problem with a stationary field by going over to a rotating frame.

As we have demonstrated above, in the quantum problem the results for the corrections to the phase and dephasing, associated with the controlled dynamics of the magnetic field, involve only the symmetric part of the noise correlator, one expects that the results for these quantities in the classical problem, expressed in terms of the noise power, would coincide with the quantum results. Indeed, we find this relation below.

Specifically, we analyze the following problem: a spin $\boldsymbol{S}$ is coupled to a controlled magnetic field $\boldsymbol{B}$ (stationary for now, but to be varied slowly in a Berry-phase experiment) and a randomly fluctuating field $\boldsymbol{X}(t)$, which we treat as a random variable with the correlation function given by $S_X(t)$. Its dynamics is governed by the Larmor equation:

$$\dot{\boldsymbol{S}} = [\boldsymbol{B} + \boldsymbol{X}(t)] \times \boldsymbol{S}\,. \tag{23}$$

This equation can be used to describe the dynamics of either a classical spin or the average spin value (i.e. the density matrix) of a spin-1/2.

As we discussed in the Introduction, we assume that the noise is weak and short-correlated, i.e., that considerable dissipative contributions to the spin dynamics arise on time scales much longer than the typical correlation time τ_{c} of the noise. Below we discuss the influence of the low- and high-frequency fluctuations on the (classical) spin dynamics

and recover the results of the quantum analysis above. Further, using the result for the low-frequency contribution we obtain the correction to the Berry phase from the environmental fluctuations at all frequencies.

3.1 Low-frequency noise: Lamb shift

Consider first the effect of a slowly fluctuating random field $\boldsymbol{X} = X\hat{z}$. Similar to the quantum-mechanical analysis in Section 2 we begin with the case of harmonic fluctuations (of random amplitude) and purely transverse noise ($\boldsymbol{B} = B\hat{x}$, i.e. $\theta = \pi/2$). Consider fluctuations $X = c_\nu \cos(\nu t)$ at a low frequency $\nu \ll B$, during a time interval δt. To evaluate the evolution operator, we analyze the dynamics in a reference frame $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ fluctuating together with the field (with the ζ-axis along $\boldsymbol{B} + \boldsymbol{X}(t)$ and the η-axis, for instance, orthogonal to $\boldsymbol{B}$ and $\boldsymbol{X}$). Since the fluctuating angular velocity of this frame's rotation is negligible, $\sim c_\nu \nu/B \ll c_\nu$, the effective magnetic field in this frame $|\boldsymbol{B}+\boldsymbol{X}(t)|\hat{\zeta}$ points along the ζ-axis. Thus the dynamics reduces to rotation about this axis by the angle $\phi(t) = \int_t^{t+\delta t} d\tau |\boldsymbol{B}+\boldsymbol{X}(\tau)| \approx \int_t^{t+\delta t} d\tau (B + X^2(\tau)/2B)$, where $B = |\boldsymbol{B}|$. Averaging the transverse spin component $S_x + iS_y \propto e^{i\phi(t)}$ one finds a lowest-order contribution to the phase factor, $\delta t \langle X^2 \rangle / 2B$, i.e. a Lamb shift $c_\nu^2/4B$ (we assumed δt much longer than the period of oscillations, $1/\nu$).

In principle, the evolution in the laboratory frame differs from that in the rotation frame. Transformation to/from the rotation frame at the beginning and the end of the time interval introduces corrections to the evolution operator or order c_ν/B. This is however a negligible boundary contribution. Indeed, for a sufficiently long time interval $\delta t \gg 1/c_\nu$ the phase shift due to the Lamb shift, of order $c_\nu^2 \delta t/B$, is much larger (but still small, as long as $\delta t \ll B/c_\nu^2$).

Similar results hold for more general low-frequency fluctuations, non-harmonic and with arbitrary direction θ. Indeed, in the same rotating frame the dynamics reduces to rotation about the ζ-axis by the angle $\phi(t) = \int_t^{t+\delta t} d\tau |\boldsymbol{B} + \boldsymbol{X}(\tau)| \approx \int_t^{t+\delta t} d\tau (B + X_\parallel(\tau) + X_\perp^2(\tau)/2B)$, where $X_\parallel = X\cos\theta$, $X_\perp = X\sin\theta$ are the longitudinal and transverse components of $\boldsymbol{X}$ (relative to $\boldsymbol{B}$). Averaging the transverse spin component $\propto e^{i\phi(t)}$ one finds, apart from dephasing, a lowest-order contribution to the phase factor, $\delta t \langle \boldsymbol{X}_\perp^2 \rangle / 2B$, and hence the Lamb shift

$$\delta E = \sin^2\theta \int \frac{d\nu}{4\pi} \frac{S_X(\nu)}{B}\,, \tag{24}$$

where θ is the angle between $\boldsymbol{B}$ and the direction $\hat{z}$ of the noise. This result coincides with the low-frequency contribution in Eq. (13).

3.2 From low frequencies to all frequencies

The expression (24) and the symmetry of the problem suggests a way to find the contribution of all, not only slow, modes in the environment to the Lamb shift (and later to the Berry phase). Indeed, we discuss weak short-correlated noise, i.e. such that its contribution to the dynamics on time scales or order τ_c is small. Contributions from different time intervals $\sim \tau_c$ are uncorrelated and add up independently. Hence in the evaluation of the (real and imaginary) contribution of such a short interval to the evolution frequencies (the Lamb shift, the dephasing and relaxation rates) it is enough to consider the lowest nonvanishing, i.e. second order.

The symmetry of the problem can be used to analyze the structure of such a second-order contribution. The spin-rotational symmetry (about the $\boldsymbol{B}$-field's direction) and the time-translational symmetry imply that (i) the longitudinal and transverse fluctuations, $X_\perp$ and $X_\parallel$, do not interfere and may be considered separately; (ii) it is convenient to expand the transverse fluctuating field in circularly polarized harmonic modes, and the latter contribute independently.

The longitudinal noise gives rise to the pure dephasing (and only the low frequencies $\lesssim 1/T_2$ contribute), without affecting the level splitting. As for the transverse noise, for a single circularly polarized mode at frequency ν it is convenient to analyze its contribution in the spin frame, rotating at frequency ν around the field $\boldsymbol{B}$. In this frame the Larmor field is $B-\nu$ in the direction of $\boldsymbol{B}$, and the fluctuating circularly polarized mode is slow. Applying to this mode Eq. (24), going back to the laboratory frame and adding up contributions of all modes we arrive at the expression for the correction to the Larmor frequency:

$$\delta E = \sin^2\theta\, \mathcal{P}\!\int \frac{d\nu}{4\pi}\frac{S(\nu)}{B-\nu}\,. \tag{25}$$

It is thus this result which needs to be compared with the quantum correction (Lamb shift) of the previous section. Symmetrization of the integral in Eq. (25) w.r.t. to ν brings it to the form of Eq. (13). Notice that regularization of this expression via the introduction of $+i0$ in the denominator allows us also to recover the imaginary part of the Lamb shift, i.e. a contribution to the dephasing rate.

3.3 High frequencies

Although Eq. (25) describes the contribution of all frequencies, it is interesting to discuss specifically the limit of high frequencies. In this subsection we provide an argument which parallels the result of

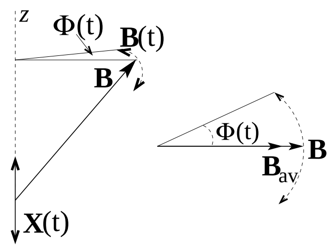

Figure 2.

subsection 2.4: the high-frequency fluctuations ($\nu \gg B$) suppress the transverse ($\perp \hat{z}$) component of the $\boldsymbol{B}$-field.

Indeed, to solve for the dynamics in the presence of high-frequency fluctuations in a fixed direction, $X(t)\hat{z}$, and the static field $\boldsymbol{B}$, let us analyze the dynamics in the frame that rotates about the $\hat{z}$-axis with angular velocity $X(t)$, i.e. differs from the lab frame by a rotation by the fluctuating angle $\Phi(t) = \int_0^t X(\tau)d\tau$. The rotation of this frame is chosen to exactly compensate for the field $X(t)\hat{z}$, and the Larmor field $\boldsymbol{B}(t)$ in this frame is just the $\boldsymbol{B}$-field, but now fluctuating due to the frame's rotation as shown in Fig. 2. The spin dynamics is governed by the Larmor equation $\dot{\boldsymbol{S}} = \boldsymbol{B}(t) \times \boldsymbol{S}$, and the value of the spin changes considerably only on time scales of order $1/B$, during which many fluctuations occur. Looking at the dynamics on intermediate time scales, between $1/\nu$ and $1/B$, one finds that the spin dynamics is governed by the value of the $\boldsymbol{B}$-field averaged over fast fluctuations. The averaging affects only the horizontal (orthogonal to z) component of the $\boldsymbol{B}$-field. The direct evaluation shows that the horizontal component is suppressed exactly by the factor $\exp[-\int\limits_0^\infty (d\nu/2\pi) S_X(\nu)/\nu^2]$ (cf. Eq. (19)). For instance, for a single mode at frequency ν we have $X(t) = 2X_\nu \cos(\nu t)$ and $\Phi(t) = 2X_\nu \sin(\nu t)/\nu$; then the transverse component of the field is suppressed by the factor $1 - \langle\Phi^2\rangle/2$, and $\langle\Phi^2\rangle/2 = \langle X_\nu^2\rangle/\nu^2$. This evaluation of the dynamics in the rotating frame relies on the small parameter B/ν.

The spin-evolution operator (before averaging) $\hat{O}_{\rm lab}(t,t')$ in the laboratory frame is related to that in the rotating frame, $\hat{O}_{\rm lab}(t,t') = \hat{O}_z(-\Phi(t))\hat{O}_{\rm rot}(t,t')\hat{O}_z(\Phi(t'))$, via the transformation $O_z(\Phi(t))$ from the lab frame to the rotating frame. However, this transformation $\hat{O}_z(\Phi(t))$ at the beginning and at the end of the evolution is close to the identity operator, and taking it into account adds only a boundary effect,

which does not grow with the size of the time interval and is therefore negligible.

3.4 Berry phase under classical noise

To find a dissipation-induced correction to the Berry phase we may use the same approach as in Section 2.3: first, we find the Lamb shift for a stationary field $\boldsymbol{B}$ and then evaluate the Berry phase using the relation (16). In this way we find the same expression (17) for the Berry phase.

4. Conclusions

In this paper we have derived expressions for the environment-induced correction to the Berry phase, for a spin coupled to an environment. On one hand, we presented a simple quantum-mechanical derivation for the case when the environment is treated as a separate quantum system. On the other hand, we analyzed the case of a spin subject to a random classical field. The quantum-mechanical derivation provides a result which is insensitive to the antisymmetric part of the random-field correlations. In other words, the results for the Lamb shift and the Berry phase are insensitive to whether the different-time values of the random-field operator commute with each other or not. This observation gives rise to the expectation that for a random classical field, with the same noise power, one should obtain the same result. For the quantities at hand, our analysis outlined above involving classical randomly fluctuating fields has confirmed this expectation.

Furthermore, we provided simple arguments, which allow one to understand the contribution of fluctuations in various frequency ranges (below and above the Larmor frequency).

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